Algebra and Number Theory Round Solutions

May 17, 2026

LAMT 2026

1. The LA Rams will play 17 games in the NFL regular season. They will play each of their three division rivals (Seahawks, Cardinals, and 49ers) two times, while playing all other teams zero or one time. If there are 32 teams (including the Rams) in the NFL, find the number of teams (excluding the Rams) that will not play the Rams during the regular season.

   Solution: $\boxed{17}$

   The Rams will play $17 - 2(3) = 11$ games against non-division opponents. In total, $11 + 3 = 14$ teams (that are not the Rams) will play the Rams. So $31 - 14 = \boxed{17}$ teams (that are not the Rams) will not play the Rams.

2. Find the unique positive real number x for which

$$\left(\frac{\sqrt{x+1}+\sqrt{x+12}}{\sqrt{x+2}+\sqrt{x+17}}\right)^2 + \frac{1}{x+2} = 1.$$

   Solution: $\boxed{\frac{7}{4} = 1.75}$

   Subtract $\frac{1}{x+2}$ from both sides and take the square root of both sides to get

$$\frac{\sqrt{x+1}+\sqrt{x+12}}{\sqrt{x+2}+\sqrt{x+17}} = \frac{\sqrt{x+1}}{\sqrt{x+2}}.$$

   Note that if $\frac{a+b}{c+d} = \frac{a}{c}$, then $\frac{b}{d} = \frac{a}{c}$. Thus $(x+12)(x+2) = (x+1)(x+17)$, or $14x + 24 = 18x + 17$, meaning $x = \boxed{\frac{7}{4}}$.

3. Brooks assigns each of the 8 positive divisors of 30 to either a row or column of the 4×4 grid below, with no two numbers being assigned to the same row or column. In each cell, he writes the least common multiple of the numbers assigned to the row and column of that cell. Find the sum of the numbers assigned to the 4 columns.

6	2	10	30
30	10	10	30
6	3	15	15
30	30	30	30

.

   Solution: $\boxed{27}$

   Unless all the 4 numbers assigned to the rows (or similarly, the columns) are divisible by some p (which is clearly not true for any of $p = 2, 3, 5$), the entry assigned to a row is divisible by p if and only if everything in that row is divisible by p. Thus for each column check each prime: if p divides everything in that column, then it divides the number assigned to that column.

   From here it is not hard to uniquely extract that the numbers assigned to the columns are 6, 1, 5, and 15. This gives an answer of $6 + 1 + 5 + 15 = \boxed{27}$.

4. Let p, q, and r be the roots of the polynomial $x^3 - 13x^2 + 40x - 3$. Find the value of

$$\frac{p^2}{1-qr} + \frac{q^2}{1-rp} + \frac{r^2}{1-pq}.$$

Solution: $\boxed{166}$

Since $pqr = 3$, by Vieta's formulas, we have

$$\frac{p^2}{1-qr} + \frac{q^2}{1-rp} + \frac{r^2}{1-pq} = \frac{p^2}{1-\frac{3}{p}} + \frac{q^2}{1-\frac{3}{q}} + \frac{r^2}{1-\frac{3}{r}},$$

which equals

$$\frac{p^3}{p-3} + \frac{q^3}{q-3} + \frac{r^3}{r-3}.$$

Then note that since they are roots, $x^3 = 13x^2 - 40x + 3$ for each of $x = p, q, r$. Thus we can substitute to get

$$\frac{13p^2 - 40p + 3}{p-3} + \frac{13q^2 - 40q + 3}{q-3} + \frac{13r^2 - 40r + 3}{r-3}.$$

Then note that we can factor

$$13x^2 - 40x + 3 = (x-3)(13x-1),$$

so the above becomes

$$\begin{aligned}(13p-1) + (13q-1) + (13r-1) &= 13(p+q+r) - 3\\ &= 13 \cdot 13 - 3\\ &= \boxed{166}.\end{aligned}$$

5. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the number of positive integers $n \le 420$ for which

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n}{5} \right\rfloor \cdot \left\lfloor \frac{n}{7} \right\rfloor$$

is even.

Solution: $\boxed{390}$

We can instead find the number of n for which this is odd, since this requires each individual term to be odd. Note that for $\left\lfloor \frac{n}{2} \right\rfloor$ to be odd we require $n \equiv 2, 3 \pmod 4$. In general for $\left\lfloor \frac{n}{k} \right\rfloor$ to be odd we require $n \pmod{2k}$ to be $\ge k$. We split this into cases.

Case 1: $n \equiv 2 \pmod 4$. This means n is even. It suffices to consider n modulo 6, 10, and 14. Modulo 6 n must be 4; modulo 10 it can be 6 or 8; and modulo 14 it can be 8, 10, or 12. This gives $1 \cdot 2 \cdot 3 = 6$ cases by the Chinese Remainder Theorem.

Case 2: $n \equiv 3 \pmod 4$. This means n is odd. Modulo 6 n can be 3 or 5; modulo 10 it can be 5, 7, or 9; and modulo 14 it can be 7, 9, 11, or 13. In total this gives $2 \cdot 3 \cdot 4 = 24$ cases.

In total there are $6 + 24 = 30$ cases where this is odd. This gives us an answer of $420 - 30 = \boxed{390}$.

6. Let a, b, and c be positive real numbers satisfying

$$\sqrt{a+1} = \sqrt{b} + \sqrt{c}, \quad \sqrt{b+2} = \sqrt{c} + \sqrt{a}, \quad \sqrt{c+3} = \sqrt{a} + \sqrt{b}.$$

Find a.

Solution: $\boxed{\frac{25}{24}}$

Let $\sqrt{a} = x$, $\sqrt{b} = y$, $\sqrt{c} = z$. Then by squaring and summing the equations we get

$$6 + x^2 + y^2 + z^2 = 2(x^2 + y^2 + z^2 + xy + yz + zx).$$

Subtracting $x^2 + y^2 + z^2$ and factoring we get

$$6 = (x + y + z)^2 \Rightarrow x + y + z = \sqrt{6}.$$

So our first equation gives us

$$\sqrt{x^2 + 1} = \sqrt{6} - x \Rightarrow x^2 + 1 = x^2 - 2\sqrt{6}x + 6.$$

From which we easily get $x = \frac{5}{2\sqrt{6}}$. Then $a = x^2 = \boxed{\frac{25}{24}}$.

7. Let a and b be positive integers. Suppose the smallest positive integer m such that b divides $am - 1$ is 81, and the smallest positive integer n such that a divides $bn - 1$ is 64. Find the smallest possible value of $a + b$.

   Solution: $\boxed{289}$

   Clearly a and b must be relatively prime for this to be possible. We see that m is the inverse of a mod b, with n defined similarly. The motivation for what follows comes from Bezout's Lemma, which hints that the inverse of a modulo b and the inverse of b modulo a are quite related to each other. In particular, if we have solved the equation

   $$ax + by = 1,$$

   for x and y, we have found both inverses.

   Consider $81a + 64b$. Taken modulo a, the $81a$ vanishes and $64b \equiv 1 \pmod{a}$. The same occurs modulo b. Thus $81a + 64b \equiv 1 \pmod{ab}$.

   The key is the utilize the size restrictions on a and b. Since 81 is the smallest inverse of a mod b, we have $81 < b$, and also $64 < a$. The quantity $81a + 64b$ is clearly positive and not equal to 1, but also

   $$81a + 64b < b \cdot a + a \cdot b < 2ab.$$

   Combining this with $81a + 64b \equiv 1 \pmod{ab}$, we see $81a + 64b = ab + 1$. From here we can easily factor and simplify, noting that $81 \cdot 64 = 72^2$:

   $$(a - 64)(b - 81) = 72^2 - 1 = 71 \cdot 73.$$

   The minimum possible value of $a + b$ will occur when $a = 64 + 71$ and $b = 81 + 73$, so the answer is $64 + 71 + 81 + 73 = \boxed{289}$.

   In fact, with 64 and 81 replaced by x^2 and y^2, the answer will be $(x + y)^2$.

8. Find the number of ordered pairs (a, b) of positive integers for which $\gcd(a, 10) = 1$, b divides 10^5, and $ab \leq 10^5$.

   Solution: $\boxed{98432}$

   Conditioned on the value of ab, there is at most one way to choose what a and b actually are. Clearly b must be the greatest common divisor of ab and 10^5, and a is whatever is left over. In this way each value of ab generates a unique pair (a, b), but for some of these values of ab the condition $\gcd(a, 10) = 1$ is not satisfied. This happens exactly for values of ab where either $\nu_2(ab) > \nu_2(10^5)$ or $\nu_5(ab) > \nu_5(10^5)$. It is easy to compute that there are

   $$\left\lfloor \frac{10^5}{5^6} \right\rfloor + \left\lfloor \frac{10^5}{2^6} \right\rfloor = 6 + 1562 = 1568$$

   values of ab with this issue. This gives us an answer of $10^5 - 1568 = \boxed{98432}$.

9. Suppose $P(x)$ and $Q(x)$ are monic quadratic polynomials for which

$$0 = P(P(4) + Q(4)) = P(P(5) + Q(5)) = P(P(9) + Q(9))$$
$$0 = Q(P(6) + Q(6)) = Q(P(8) + Q(8)) = Q(P(9) + Q(9)).$$

Find $P(0) + Q(0)$.

Solution: $\boxed{96}$

Let $R(x) = P(x) + Q(x)$. Then note that $P(R(x))$ and $Q(R(x))$ both have the same axis of symmetry as $R(x)$. The only possible axis of symmetry is at $x = 7$, which means $x = 10$ and $x = 5$ are roots of $P(R(x)$ and $Q(R(x))$, respectively. We also get that $R(x) = 2(x - 7)^2 + a$ for some a.

Then the value of $R(4) = R(10) = 18 + a$, $R(5) = R(9) = 8 + a$, and $R(6) = R(8) = 2 + a$. These are the roots of P and Q. The sum of the roots of P and Q is -1 times the x coefficient of R, which is -28. This means

$$(18 + a) + (8 + a) + (2 + a) + (8 + a) = 28,$$

so $a = -2$. Since P has roots $18 + a$ and $8 + a$, we get $R(x) = 2(x - 7)^2 - 2$, meaning $R(0) = \boxed{96}$.

10. Find the largest integer $n < 1000$ such that

$$\left|\sum_{k=1}^{n} ki^k\right|$$

is an integer.

Solution: $\boxed{238}$

Consider the sum in cases of n modulo 4:

- $n \equiv 0 \pmod 4$:
$$\sum_{k=1}^{n} ki^k = \frac{n}{4}(2 - 2i) = \frac{n}{2}(1 - i).$$
- $n \equiv 1 \pmod 4$:
$$\sum_{k=1}^{n} ki^k = \frac{n-1}{4}(2 - 2i) + ni = \frac{n-1}{2} + \frac{n+1}{2}i.$$
- $n \equiv 2 \pmod 4$:
$$\sum_{k=1}^{n} ki^k = \frac{n-2}{4}(2 - 2i) + (n-1)i - n = \frac{-2-n}{2} + \frac{n}{2}i.$$
- $n \equiv 3 \pmod 4$:
$$\sum_{k=1}^{n} ki^k = \left(\sum_{k=1}^{n+1} ki^k\right) - (n+1) = \frac{n+1}{4}(2 - 2i) - (n+1) = -\frac{n+1}{2} - \frac{n+1}{2}i.$$

For the $n \equiv 0, 3 \pmod 4$ case, the magnitude will always be a multiple of $\sqrt{2}$, so we ignore these cases. For the $n \equiv 1 \pmod 4$, the magnitude is $\sqrt{\frac{n^2+1}{2}}$. For the $n \equiv 2 \pmod 4$, the magnitude is $\sqrt{\frac{n^2+2n+2}{2}} = \sqrt{\frac{(n+1)^2+1}{2}}$.

Both of the above cases can be resolved together by considering the Pell equation $k_1^2 - 2k_2^2 = -1$. If (k_1, k_2) is a solution, then we could possibly plug in $n = k_1$ or $n = k_1 - 1$ to yield an integer value for the magnitude of our sum.

We can find solutions to $k_1^2 - 2k_2^2 = -1$ by first finding the fundamental solution. We get $(1, 1) = (a_0, b_0)$ is the fundamental solution, and the following recursion can generate successive solutions:

$$a_{n+1} + b_{n+1}\sqrt{2} = (1 + \sqrt{2})^2(a_n + b_n\sqrt{2}) = (3 + 2\sqrt{2})(a_n + b_n\sqrt{2}).$$

Thus, using this algorithm, we yield the sequence: $1+\sqrt{2}$, $7+5\sqrt{2}$, $41+29\sqrt{2}$, $239+169\sqrt{2}\ldots$. We stop caring after the listed term, since the integer part of the next number in our sequence is well over 1000. Noting that $238 \equiv 2 \pmod 4$ and $\frac{(238+1)^2+1}{2} = 169^2$ (resulting from our recursion), our answer is $\boxed{238}$.

11. **[TIEBREAKER]** Let $(a_1, a_2, \ldots, a_9)$ be a permutation of $(1, 2, \ldots, 9)$. Define the function

$$f(x) = |\ldots||x-a_1|-a_2|-a_3|\ldots|-a_9|.$$

Estimate the maximum possible area bounded by the graph of $f(x)$ and the x-axis on the interval $[0, 45]$, over all permutations of $(1, 2, \ldots, 9)$.

Express your answer as a number in base 10 (submissions not in this form will not be accepted). Ties will be broken based on distance to the correct answer.

Solution: $\boxed{346.5}$